Hyperfinite graphings, part II

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Theorem (Bowen–Kun–S.)

Any bipartite hyperfinite a.e. one-ended regular graphing admits a **measurable perfect matching**.

Example (Laczkovich)

There exists a 2-regular (so bipartite, two-ended, hyperfinite) graphing that **does not admit a measurable perfect matching**.

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Proof

Consider an irrational rotation $T_\theta : S^1 \to S^1$ and let G be the Schreier graph of the induced $\mathbb{Z}$-action.

Definition

A measurable map $T : (V, \nu) \to (V, \nu)$ is **ergodic** if any T-invariant measurable set is either measure 1 or 0.

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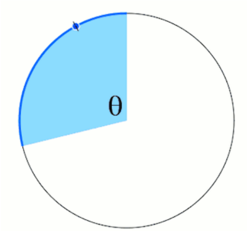
Fact

Any **irrational rotation is ergodic**.



Proof continued

Since θ is irrational, both T_θ and $T_{2\theta}$ are ergodic.

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Suppose M is a measurable perfect matching. Let

$$A = \{x \in S^1 : (x, T_\theta(x)) \in M\}.$$

Note that A is $T_{2\theta}$-**invariant**, so either null or co-null. Note that

$$B = S^1 \setminus A$$

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But $T_\theta(A) = B$ and T_θ **preserves the measure**. Contradiction.

Treeings

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Spanning treeings

Given a graphing G, by a **spanning treeing** we mean a treeing contained in G whose connected components are the same as those of G.

Fact

Any hyperfinite graphing admits a (measurable) **spanning treeing**.

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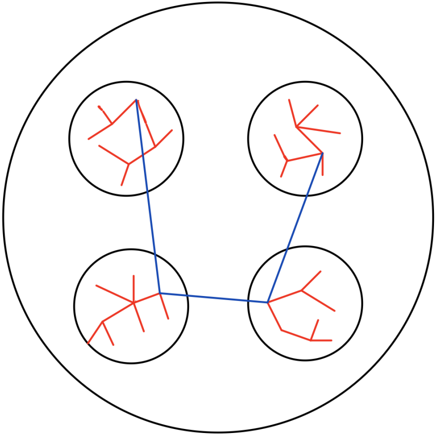
Proof

Any **finite graph admits a spanning tree** and we can use the finite graphs approximating the graphing to construct a spanning treeing.

Proof by picture



Finite tree

Recall that a finite tree with $v = |V|$ vertices has $|E| = v - 1$ edges.

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Average degree

Since

$$\sum_{x \in V} \deg(x) = 2|E|,$$

we get that the **average degree in a finite tree** is equal to

$$\frac{1}{v} \sum_{x \in V} \deg(x) = \frac{2|E|}{|V|} = 2\frac{v-1}{v} \xrightarrow[v \to \infty]{} 2$$

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The **cost** of any spanning treeing of a hyperfinite graphing is 1.

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Average degree

This means that the **average degree** of a spanning treeing of a hyperfinite graphing **is equal to** 2.

Convex analysis on fractional perfect matchings

Suppose G is a bipartite regular graphing. We consider the set

$$C_G = \{\varphi \in L^2(E(G)) : \varphi \text{ is a fractional perfect matching}\}$$

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Extreme points

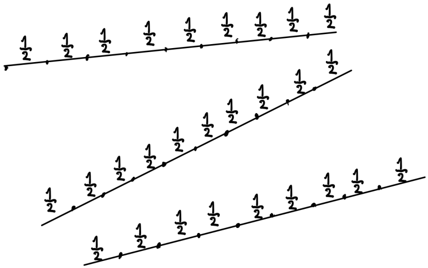
By the **Krein–Milman theorem**, C_G has an extreme point (if it is nonempty).

Fact

If φ is an extreme point of C_G, then for a.e. edge $e \in E(G)$ we have

$$\varphi(e) \in \{0, \frac{1}{2}, 1\}$$

and the **set of edges on which** $\varphi = \frac{1}{2}$ **is a disjoint union of lines**, which we denote by $L(\varphi)$.



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Claim

Suppose φ is an extreme point of C_G. The set

$$F = \{e \in E(G) : 0 < \varphi(e) < 1\}$$

is a subtreeing (i.e. has no cycles)

Proof of Claim

Suppose the set of edges in F that lie in a cycle contained in F has positive measure. We can refine F to a **positive measure subset that consists of edge-disjoint cycles** and assume for some $\varepsilon > 0$, on every $e \in F$ we have

$$\varepsilon < f(e) < 1 - \varepsilon.$$

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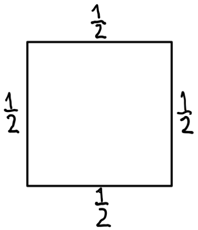
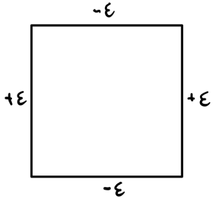
Recall that each cycle has even length.



Note that we can **add** ε **on the even edges** of the cycles and **subtract** ε **on the odd edges**. Write φ_+ for this fractional perfect matching.



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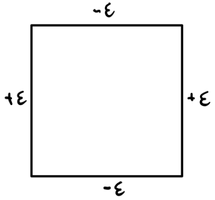
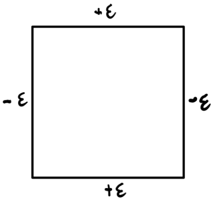


Note that we can also **subtract** ε **on the even edges** of the cycles and **add** ε **on the odd edges**. Write φ_- for this fractional perfect matching.



But now

$$\varphi = \frac{\varphi_{+\varepsilon} + \varphi_{-\varepsilon}}{2},$$

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This ends the proof of the Claim.

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By the previous claim, the set F **is acyclic, so is a subtreeing**. Since F is contained in G, it is **hyperfinite**.

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As the average degree of F is 2 and F has no leaves, we get that **a.e. vertex in the graphing spanned by F must have degree** 2, which means that F is a disjoint union of lines.

Step 3

We claim that on a.e. line in F the fractional perfect matching φ equals to $\frac{1}{2}$.

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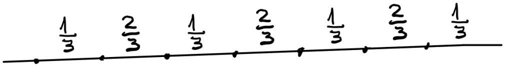
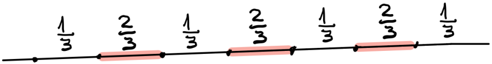
Suppose it is not the case and we have a **positive measure set of lines** in F on which

$$\varphi \neq \frac{1}{2}$$

and $\varphi > \varepsilon$ and $\varphi < 1 - \varepsilon$.



Note that if $\varphi \neq \frac{1}{2}$ on a line l, then we can **call an edge even if** $\varphi(e) > \frac{1}{2}$ and odd otherwise.



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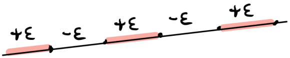

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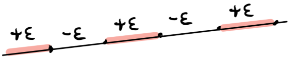

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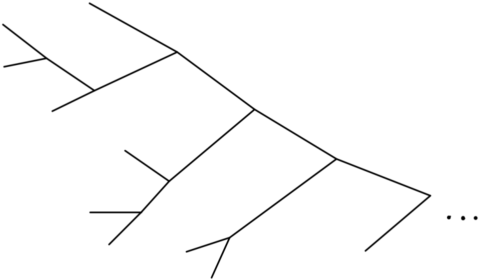
This ends the proof of the Fact describing extreme points of C_G.

One-ended trees

A **one-ended tree** is a tree which has one end (as a graph).



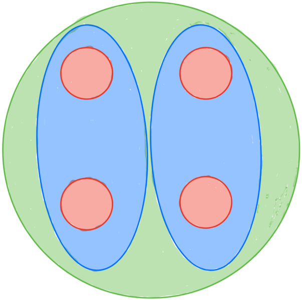
Theorem (Gaboriau–Conley–Marks–Tucker-Drob)

Any hyperfinite one-ended graphing G admits a **one-ended spanning treeing**, i.e. a subgraphing T such that the components of T are one-ended trees and are the same as the components of G.

Toast

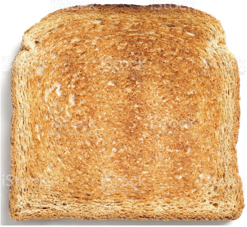
Given a Borel graph G, we say that a Borel collection $\mathcal{T}$ of finite connected subsets of $V(G)$ is a **toast** if it satisfies:

- $\bigcup_{K\in\mathcal{T}} E(K) = E(G)$,
- for every pair $K, L \in \mathcal{T}$
  - either $(K \cup N(K)) \cap L = \emptyset$
  - or $K \cup N(K) \subseteq L$,
  - or $L \cup N(L) \subseteq K$,



Toast

Given a Borel graph G, we say that a Borel collection $\mathcal{T}$ of finite connected subsets of $V(G)$ is a **connected toast** if it is a toast and additionally satisfies:



- for every $K \in \mathcal{T}$ the induced subgraph on $K \setminus \bigcup_{K \supsetneq L \in \mathcal{T}} L$ is connected.

Lemma

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Proof

We use a one-ended spanning treeing and construct the **toast along the treeing**.